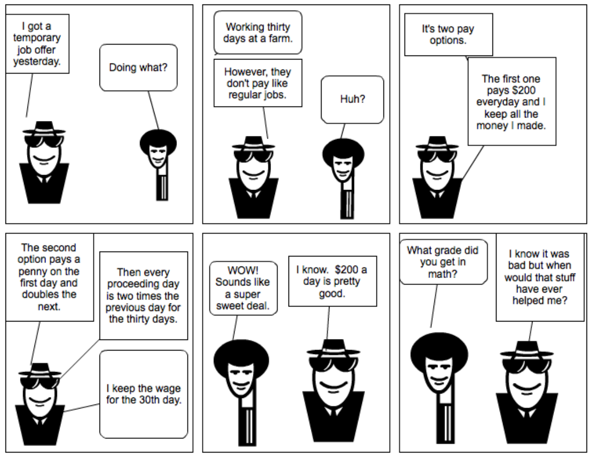
Unit 9 Notes
Exponential Functions



Tentative Schedule

Day	Classwork	Assignment
Fri. 2/6 Mon. 2/9	Quest #8	Video #9.1 – Integer Sequences and Recursive Formulas
Tues. 2/10	1 – 24	Video #9.2 – Arithmetic and Geometric Sequences
Wed. 2/11 Thurs. 2/12	25 – 40 (41 – 43 are optional)	Catch-up Time (**make sure you understand all kinds of sequences and their notation**)
Fri. 2/13	Class discussion of notation of sequences	Review sheet on sequences
Mon. 2/23 Tues. 2/24	Class discussion of exponential growth	Video #9.4 - Simple and Compound Interest
Wed. 2/25	44 – 51	Video #9.5 – Exponential Decay
Thurs. 2/26 Fri. 2/27	52 – 60 **Take-Home Quiz Due**	Correct Practice Packet
Mon. 3/2	Review for Test #9	Review for Test #9
Tues. 3/3 Wed. 3/4	**Test #9**	Video #10.1 with Notes – Solving Quadratic Equations by Factoring

Name: ____

Notes 9.1 - Integer Sequences and Recursive Formulas

1. Consider the sequence that follows a "plus 3" pattern: 4, 7, 10, 13, 16…

f(1) f(2) f(3) f(4) f(5)

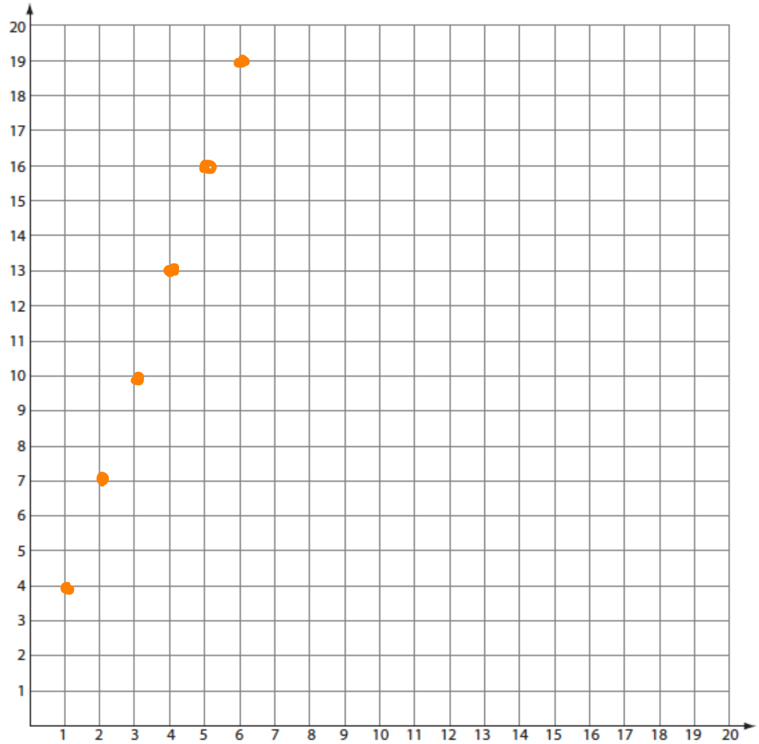
therefore f(0) = 1

A. Write a formula for the sequence using both a_n notation and the $f(n)$ notation.

$f(n) = 3n + 1$

increases by 3 (m = 3)

starts @ 1



B. Does the formula $f(n) = 3(n-1) + 4$ generate the same sequence?

$f(n) = 3(n-1) + 4$

$f(n) = 3n - 3 + 4$

$f(n) = 3n + 1$

Yes!

C. Graph the terms of the sequence as ordered pairs on the coordinate plane. What do you notice about the graph?

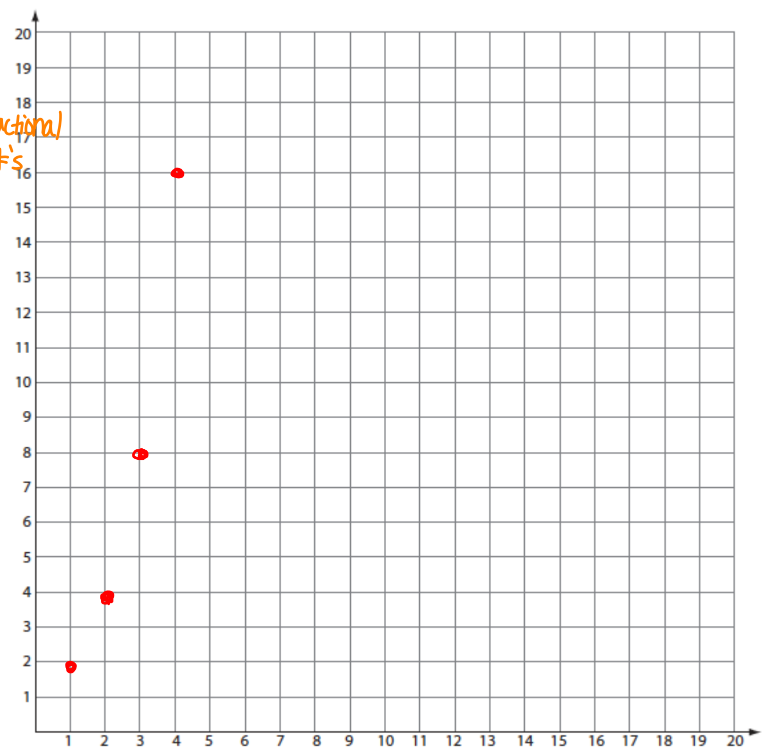
Linear!

(Don't connect dots for sequences – there aren't fractional term #'s)

2. Justin has thought of a pattern that shows powers of two. Here are the first 6 numbers in Justin's sequence:
2, 4, 8, 16, 32, 64,…
Write an expression for the nth number of Justin's sequence.

$f(n) = 2^n$

Exponential!



Explicit Formula	**Recursive Formula**
tells any term in the sequence	tells the next term in a sequence

1. Consider the sequence 5, 8, 11, 14, 17, …

   a. What do you think is the next number in the sequence?

   20 - add 3 for each term

   b. What is the **explicit** formula for this sequence?

   Linear
   $m = 3$ (+3 each term)
   $b = 2$ ($f(0) = 2$)

   $\boxed{f(n) = 3n+2}$

   c. Write a **recursive** formula for this sequence.

   $\underbrace{f(n+1)}_{\text{next term}} \underbrace{=}_{\text{is}} \underbrace{f(n)}_{\text{current term}} \underbrace{+3}_{\text{plus 3}}$ for $\underbrace{f(1)=5}_{\text{first term is 5}}$

2. What would happen if we were to change the "+" sign in the recursive formula to a "-" sign or a "x" sign or a "÷" sign?

   a. What sequence does $A(n+1) = A(n) - 3$ for $n \ge 1$ and $A(1) = 5$ represent?

   $5 - 3 = 2$
   $2 - 3 = -1$
   $-1 - 3 = -4$
   $-4 - 3 = -7$

   $\boxed{5, 2, -1, -4, -7, \ldots}$

   b. What sequence does $a_{n+1} = a_n \bullet 3$ for $n \ge 1$ and $a_1 = 5$ represent?

   $5 \cdot 3 = 15$
   $15 \cdot 3 = 45$
   $45 \cdot 3 = 135$
   $135 \cdot 3 = 405$

   $\boxed{5, 15, 45, 135, 405, \ldots}$

   c. What sequence does $a_{n+1} = a_n \div 3$ for $n \ge 1$ and $a_1 = 5$ represent

   $5 \div 3 = 5/3$
   $\frac{5}{3} \div 3 = \frac{5}{9}$
   $\frac{5}{9} \div 3 = \frac{5}{27}$
   $\frac{5}{27} \div 3 = \frac{5}{81}$

   $\boxed{5, \frac{5}{3}, \frac{5}{9}, \frac{5}{27}, \frac{5}{81}, \ldots}$

3. Consider the sequence 7, 11, 15, 19, 23, 27, …

   a. Write an explicit formula for the sequence.

   Linear
   $m = 4$ (increases by 4 each term)
   $b = 3$ ($f(0) = 3$)

   $\boxed{f(n) = 4n + 3}$

   b. Write a recursive formula for the sequence.

   $\underbrace{f(n)}_{\text{current term}} = \underbrace{f(n-1)}_{\text{previous term}} \underbrace{+4}_{\text{plus 4}}$ for $\underbrace{f(1) = 7}_{\text{first term is 7}}$ (slightly different than 1c, but both work)

4. Consider the sequence 3, 9, 27, 81, …
   3^1 3^2 3^3 3^4 … 3^n

   a. Write an explicit formula for the sequence.

   $\boxed{f(n) = 3^n}$

   b. Write a recursive formula for the sequence.

   $\underbrace{a_{n+1}}_{\text{next term}} = \underbrace{a_n}_{\text{current term}} \underbrace{\cdot 3}_{\text{multiplied by 3}}$ for $\underbrace{a_1 = 3}_{\text{1st term is 3.}}$

Notes 9.2 - Arithmetic and Geometric Sequences

General Formulas:

	Arithmetic Sequences	Geometric Sequences
Recursive Formula don't forget to define a_1	$a_{n+1} = a_n + d$ $(d = a_{n+1} - a_n)$ d = common difference	$a_{n+1} = a_n \cdot r$ $\left(r = \frac{a_{n+1}}{a_n}\right)$ r = Common ratio
Explicit Formula	$a_n = a_1 + d(n-1)$ OR $a_n = a_0 + dn$	$a_n = a_1 \cdot r^{n-1}$ $a_n = a_0 \cdot r^n$

Sequence	Type of Sequence	Pattern	Formula
-2, 2, 6, 10, …	Arithmetic	$10-6=4$ $6-2=4$ $2-{}^{-}2=4$ $d=4$	Recursive: $a_{n+1} = a_n + 4$ for $a_1 = -2$ Explicit: $a_n = -2 + 4(n-1)$ $a_n = -2 + 4n - 4$ $a_n = 4n - 6$
2, 4, 8, 16, … $a_0 = \frac{2(a_1)}{2(r)} = 1$	Geometric	$\frac{16}{8} = 2$ $\frac{8}{4} = 2$ $\frac{4}{2} = 2$ $r = 2$	Recursive: $a_n = a_{n-1} \cdot 2$ for $a_1 = 2$ Explicit: $a_n = 2 \cdot 2^{n-1}$ or $a_n = 1 \cdot 2^n$
$1, \frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \ldots$ $f(0) = 1 \div \left(\frac{1}{3}\right) = 3$	Geometric	$\frac{1}{27} \div \frac{1}{9} = \frac{1}{27} \cdot \frac{9}{1} = \frac{1}{3}$ $\frac{1}{9} \div \frac{1}{3} = \frac{1}{9} \cdot \frac{3}{1} = \frac{1}{3}$ $\frac{1}{3} \div 1 = \frac{1}{3}$ $r = \frac{1}{3}$	Recursive: $f(n+1) = f(n) \cdot \frac{1}{3}$ for $f(1) = 1$ Explicit: $f(n) = 1 \cdot \left(\frac{1}{3}\right)^{n-1}$ OR $f(n) = 3 \cdot \left(\frac{1}{3}\right)^n$
$\frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}, \ldots$ $a_0 = \frac{1}{2} - \frac{1}{2} = 0$	Arithmetic	$1 - \frac{1}{2} = \frac{1}{2}$ $\frac{3}{2} - 1 = \frac{1}{2}$ $2 - \frac{3}{2} = \frac{1}{2}$ $\frac{5}{2} - 2 = \frac{1}{2}$ $d = \frac{1}{2}$	Recursive: $a_n = a_{n-1} + \frac{1}{2}$ for $a_1 = \frac{1}{2}$ Explicit: $\boxed{a_n = \frac{1}{2}n}$
4, -1, -6, -11 $f(0) = 4 - {}^{-}5 = 9$	Arithmetic	$-1-4=-5$ $-6-{}^{-}1=-5$ $-11-{}^{-}6=-5$ $d=-5$	Recursive: $f(n) = f(n-1) - 5$ for $f(1) = 4$ Explicit: $f(n) = -5n + 9$
10, 1, 0.1, 0.01, 0.001… $a_0 = 10 \div (0.1) = 100$	Geometric	$\frac{.001}{.01} = .1$ $\frac{.01}{.1} = .1$ $\frac{.1}{1} = .1$ $\frac{1}{10} = .1$ $r = 0.1$	Recursive: $a_{n+1} = a_n \cdot (0.1)$ for $a_1 = 10$ Explicit:

$a_n = 10(0.1)^{n-1}$ OR $a_n = 100(0.1)^n$

Notes 9.3 - The Power of Exponential Growth

1. Two equipment rental companies have different penalty policies for returning a piece of equipment late:
Company 1: On day 1, the penalty is \$5. On day 2, the penalty is \$10. On day 3, the penalty is \$15. On day 4, the penalty is \$20 and so on, increasing by \$5 each day the equipment is late.
Company 2: On day 1, the penalty is \$0.01. On day 2, the penalty is \$0.02. On day 3, the penalty is \$0.04. On day 4, the penalty is \$0.08 and so on, doubling in amount each additional day late.
Jim rented a digger from Company 2 because he thought it had the better late return policy. The job he was doing with the digger took longer than he expected, but it did not concern him because the late penalty seemed so reasonable. When he returned the digger 15 days late, he was shocked by the penalty fee. What did he pay, and what would he have paid if he had used Company 1 instead?

Company 1	
Day	Penalty
1	5
2	10
3	15
4	20
5	25
6	30
7	35
8	40
9	45
10	50
11	55
12	60
13	65
14	70
15	75

Company 2	
Day	Penalty
1	0.01
2	0.02
3	0.04
4	0.08
5	0.16
6	0.32
7	0.64
8	1.28
9	2.56
10	5.12
11	10.24
12	20.48
13	40.96
14	81.92
15	163.84

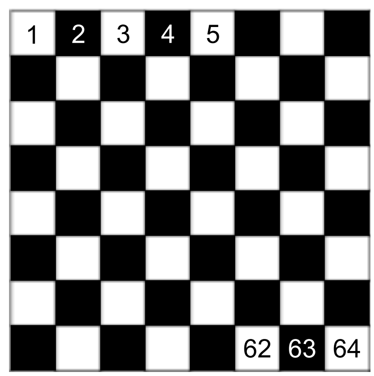
2. Folklore suggests that when the creator of the game of chess showed his invention to the country's ruler, the ruler was highly impressed. He was so impressed, he told the inventor to name a prize of his choice. The inventor, being rather clever, said he would take a grain of rice on the first square of the chessboard, two grains of rice on the second square of the chessboard, four on the third square, eight on the fourth square, and so on, doubling the number of grains of rice for each successive square. The ruler was surprised, even a little offended, at such a modest price, but he ordered his treasurer to count out the rice.

a. Why is the ruler surprised? What makes him think the inventor requested a "modest price"?

He thinks the price is low, while it's not at all! ☺

b. The treasurer took more than a week to count the rice in the ruler's store, only to notify the ruler that it would take more rice than was available in the entire kingdom. Shortly thereafter, as the story goes, the inventor became the new king. Imagine the treasurer counting the needed rice for each of the 64 squares. We know that the first square is assigned a single grain of rice, and each successive square is double the number of grains of rice of the former square. The following table lists the first five assignments of grains of rice to squares on the board. How can we represent the grains of rice as exponential expressions?

Square #	Grains of Rice	Exponential Expression
1	1	2^0
2	2	2^1
3	4	2^2
4	8	2^3
5	16	2^4



c. Write the exponential expression that describes how much rice is assigned to each of the last three squares of the board.

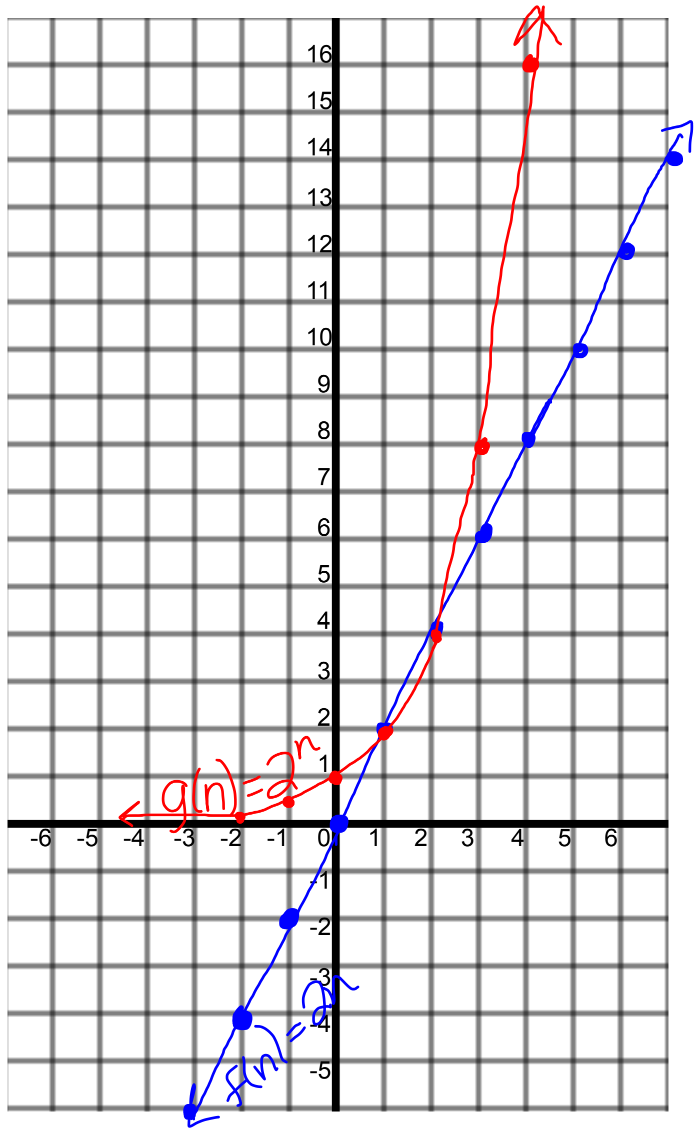
Square #	Exponential
62	2^{61}
63	2^{62}
64	2^{63}

3. Let us understand the difference between $f(n) = 2n$ and $f(n) = 2^n$.

a. Complete the tables below, and then graph the points $(n, f(n))$ on a coordinate plane for each of the formulas.

n	$f(n) = 2n$
-2	-4
-1	-2
0	0
1	2
2	4
3	6

n	$g(n) = 2^n$
-2	$2^{-2} = \frac{1}{2^2} = \frac{1}{4}$
-1	$2^{-1} = \frac{1}{2^1} = \frac{1}{2}$
0	$2^0 = 1$
1	$2^1 = 2$
2	$2^2 = 4$
3	$2^3 = 8$



b. Describe the change in each sequence when n increases by 1 unit for each sequence.

$f(n) = 2n$

Rate of change is a common difference of 2

$g(n) = 2^n$

Rate of change is a common ratio of 2

4. You invest \$5000 in a bank that offers an interest rate of 4% compounded annually. Find out how much money you would have in the bank after:

a. one year

$5000(1.04) = \boxed{\$5200}$

b. four years

1: $5000(1.04) = \$5200$

2: $5200(1.04) = \$5408 \rightarrow (5000)(1.04)(1.04) = 5000(1.04)^2$

3: $5408(1.04) = \$5624.32 \rightarrow (5000)(1.04)(1.04)(1.04) = 5000(1.04)^3$

4: $5624.32(1.04) = \$5849.29 \rightarrow (5000)(1.04)(1.04)(1.04)(1.04) = 5000(1.04)^4$

c. twenty years

20

$5000(1.04)^{20} \approx \boxed{\$10{,}955.62}$

d. Write an explicit formula to represent how much money you get in t years.

$$\boxed{f(t) = 5000(1.04)^t}$$

5. A typical thickness of toilet paper is 0.001 inches. Seems pretty thin, right? Let's see what happens when we start folding the toilet paper.
   a. How thick is the stack of paper after 1 fold? After 2 folds? After 5 folds?

$1: (0.001)(2) = 0.002 \rightarrow (0.001)(2)^1$

$2: (0.002)(2) = 0.004 \rightarrow (0.001)(2)(2) = 0.001 \cdot 2^2$

$3: (0.004)(2) = 0.008 \rightarrow (0.001)(2)(2)(2) = 0.001 \cdot 2^3$

$5: 0.001(2)^5 =$

$\boxed{0.032''}$

   b. Write an explicit formula for the sequence that models the thickness of the folded toilet paper after n folds.

$\boxed{f(n) = 0.001 \cdot 2^n}$

   c. After how many folds will the stack of folded toilet paper pass the 1 foot mark? $= 12''$

$f(13) = 8.192''$

$f(14) = 16.384''$

$\boxed{14 \text{ folds}}$

   d. The moon is about 240,000 miles from Earth. Compare the thickness of the toilet paper folded 50 times to the distance from Earth.

$f(50) = 0.001(2)^{50}$

$f(50) \approx 1{,}125{,}899{,}906{,}843''$

$1{,}125{,}899{,}906{,}843 \text{ in} \cdot \frac{1 \text{ ft}}{12 \text{ in}} \cdot \frac{1 \text{ mi}}{5280 \text{ ft}} = 17{,}769{,}884$ miles

You can go to the moon and back 37 times and thickness would be greater! ☺

Notes 9.4 - Interest

Simple Interest	Compound Interest	Exponential Growth
$I = Prt$ — I = interest, P = principal, r = rate, t = time $f(t) = P + Prt$	$f(t) = P(1+r)^t$	$f(t) = a \cdot r^t$ a = initial amount r = common ratio or growth factor

6. Peter needs $200 to start a snow cone stand for this hot summer. He borrows the money from a bank that charges 4% simple interest a year.
   a. How much will he owe if he waits 1 year to pay back the loan?

   $P = 200$, $r = 0.04$, $t = 1$ $\quad I = Prt \rightarrow I = 200(.04)(1) = \$8$

   b. How much will he owe if he waits 3 years to pay back the loan?

   1: $200(.04) = \$8$
   2: $8 + 8 = \$16$
   3: $8 + 8 + 8 = \$24$
   OR $\quad I = 200(.04)(3) = \$24$

   c. Write a formula for the amount he will owe after t years.

   $I = 400(.04)t \quad \boxed{I = 8t}$

7. Jack has $500 to invest. The bank offers an interest rate of 6% compounded annually.
   a. Write a formula for the amount he will owe after t years.

   $P = 500$, $r = .06$ $\quad J(t) = 500(1+.06)^t \quad \boxed{J(t) = 500(1.06)^t}$

   b. How much will he owe if he waits 1 year to pay back the loan?

   $J(1) = 500(1.06)^1$
   $\boxed{J(1) = \$530}$

   c. How much will he owe if he waits 3 years to pay back the loan?

   $J(3) = 500(1.06)^3$
   $\boxed{J(3) \approx \$595.51}$

8. If you have $200 to invest for 10 years, would you rather invest your money in a bank that pays 7% simple interest or 5% interest compounded annually? $P = 200$ $\quad r_1 = .07$ (simple) $\quad r_2 = .05$ (compound) $\quad t = 10$

   Simple:
   $S(t) = P + Prt$
   $S(10) = 200 + 200(.07)(10)$
   $S(10) = \$340$

   Compound:
   $C(t) = P(1+r)^t$
   $C(10) = 200(1+.05)^{10}$
   $C(10) = 200(1.05)^{10} \approx \$325.79$

   For 10 yrs – Simple @ 7%. If longer – compound

9. A rare coin appreciates at a rate of 5.2% per year. If the initial value of the coin is $500, after how many years will its value cross the $3,000 mark? Show the formula that will model the value of the coin after t years.

   $P = 3000$
   $r = 0.052$
   $t = ?$

   $C(t) = 500(1+.052)^t$
   $C(t) = 500(1.052)^t$

   Guess + check!
   You'll get $C(36) \approx \$3101.25$
   $\boxed{36 \text{ yrs}}$

Notes 9.5 - Exponential Decay

1. Jon bought a new car for $15,000. As he drove it off the lot, his friend, Riley, told him that the car's value just dropped by 15% and that it would continue to depreciate 15% of its current value each year. If the car's value is now $12,750 (according to Will), what will its value be after 5 years?

   a. Complete the table below to determine the car's value after each of the next five years.

Number of years, t, passed since driving the car off lot	Car value after t years	15% depreciation of current car value	Car value minus the 15% depreciation
0	$12,750.00	$1,912.50	$10,837.50
1	10,837.50	$1625.63	$9,211.87
2	$9,211.75	$1,381.78	$7,829.97
3	$7,829.97	$1,174.50	$6,655.47
4	$6,655.47	$998.32	$5,657.15
5	$5,657.15	$848.57	$4,808.58

   b. Write an explicit formula for the sequence that models the value of Jon's car t years after driving it off the lot.

   $C(t) = 10837.50(1-.15)^t$ $\boxed{C(t) = 10837.50(0.85)^t}$

   c. Use the formula from part (b) to determine the value of Jon's car five years after its purchase. Round your answer to the nearest cent. Compare the value with the value in the table. Are they the same?

   $C(5) = 10837.50(.85)^5 \approx 4808.66$

   Not the exact same, but close due to rounding errors

   d. Use the formula from part (b) to determine the value of Malik's car 7 years after its purchase. Round your answer to the nearest cent.

   $C(7) = 10837.50(.85)^7 \approx \boxed{\$3,474.25}$

   e. **How is exponential decay like exponential growth?**

   You're still multiplying by a "growth" factor → when $r > 1$, growth → when $0 < r < 1$, decay

2. Identify the initial value in each formula below, and state whether the formula models exponential growth or exponential decay. Justify your response.

   a. $f(t) = 2\left(\frac{2}{5}\right)^t$

   initial: 2
   decay $r = \frac{2}{5} < 1$

   b. $f(t) = \frac{2}{3}\left(\frac{1}{3}\right)^t$

   initial $= \frac{2}{3}$
   decay. $r = \frac{1}{3} < 1$

   c. $f(t) = \frac{2}{3}(3)^t$

   initial $= \frac{2}{3}$
   growth : $r = 3 > 1$

d. $f(t) = 2\left(\frac{5}{3}\right)^t$

initial = 2
growth: $r = \frac{5}{3} > 1$

e. $f(t) = \frac{3}{2}\left(\frac{2}{3}\right)^t$

initial = $\frac{3}{2}$
decay → $r = \frac{2}{3} < 1$

3. Kelli's mom takes a 400 mg dose of aspirin. Each hour, the amount of aspirin in a person's system decreases by about 29%. How much aspirin is left in her system after 6 hours?

$P = 400 \quad r = 0.29 \quad t = 6$

$f(t) = 400(1 - 0.29)^t$

$f(6) = 400(.71)^6 \approx \boxed{51\text{ g}}$

4. According to the International Basketball Association (FIBA), a basketball must be inflated to a pressure such that, when it is dropped from a height of 1,800 mm, it will rebound to a height if 1,300 mm. Luke decides to test the rebound-ability of his new basketball. Let $f(n)$ be the height of the basketball after n bounces.

a. Assume the ratio of each rebound height to the previous rebound height remains the same. What is the ratio?

$\frac{1300}{1800} = \frac{13}{18}$

b. Complete the table below to reflect the heights Luke expects to measure.

n	$f(n)$	
0	1,800	
1	1,300	
2	939	a)
3	678	b)
4	490	c)

a) $1300 \cdot \frac{13}{18} \approx 939$

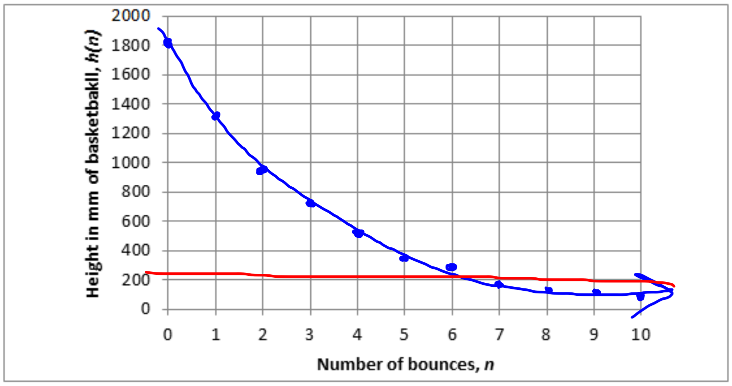
b) $939 \cdot \frac{13}{18} \approx 678$

c) $678 \cdot \frac{13}{18} \approx$ 490

c. Write the explicit formula for the sequence that models the height of the basketball after any number of bounces.

$$\boxed{f(t) = 1800\left(\frac{13}{18}\right)^t}$$

d. Plot the points from the table. Connect the points with a smooth curve.



e. Use the curve to estimate the bounce number at which the rebound height will drop below 200 mm.

Around the 7th bounce.